Abstract—Dynamic response of a structural system in frequency domain can be calculated by summing the frequency response of dynamic sub-models. However depending on the frequency content of external disturbance, some dynamic sub-models might be more active; these dynamic sub-models can be identified by calculating the cost-to-go to return to the origin. These cost-to-go to return to the origin for each dynamic sub-model was calculated by solving Lyapunov equation. Then comparing these cost-to-go to return to the origin, a novel switching rule was designed to switch to the dynamic sub-model with the highest cost-to-go to return to the origin. This particular dynamic sub-model was regulated by the associated modal controller. Thus it has been shown that dynamic sub-model with the highest cost-to-go to return to the origin at any time can be controlled instead of controlling all modes of motion of the structural system and the consequence of this switching control concept is minor from the controller perspective.

Keywords—Active vibration control, switching multiple model control, modal control.

I. INTRODUCTION

In frequency domain, it is known that response of a structural system can be calculated by summing response of dynamic sub-models which are represented by the frequency response of single-degree-of-freedom systems. Since the disturbance frequency content is usually unknown, it can be considered as an uncertain parameter which can take any possible value in the spectrum of structural system natural frequencies. When the external disturbance is in the neighborhood of any natural frequency of the structural system, the single-degree-of-freedom dynamic sub-model associated with that natural frequency would be dominant. It is clear that the mathematical model of a structural system behaves like switching among the dynamic sub-models with respect to the unknown frequency content of the disturbance.

A close look at the system response in the frequency domain would reveal that not every dynamic sub-model has equal importance from the point of view of controlling system response: controlling a subset of dynamic sub-models may be quite sufficient to meet the optimality criteria of the controller objective. This observation has led to the conclusion that a multiple model control scheme can be used to control the most active dynamic sub-model of the structural system and this might be quite sufficient from the controller point of view.

This article claims that any structural system can be represented by a multiple model switching systems where the models represent the dynamic sub-models associated with each mode of motion. The main goal of this article is to show that any dynamic sub-model of the structural system can be controlled and that controlling a single dynamic sub-model is equivalent to controlling the whole system. Besides, an optimal switching rule was proposed to switch to the modal controller that regulates the most active dynamic sub-model. It has been shown that this switching rule can be designed by solving the Lyapunov equation.

The rest of the paper is organized as follows: In section two, the related works are reviewed; the contribution of this study to contemporary literature is explained. This section is concluded with a discussion of the current state of the art of the related works. In section three, problem statement is presented. Section four is dedicated to the theory where the main contribution of this study is explained in detail. In section five, numerical results are presented; performance of the switching modal control is examined. Section six presents the conclusions with a focus on the key contribution of this study to the contemporary literature on related field.

II. RELATED WORKS

In the design of multiple model control there are two issues: i) design of model set and ii) design controller set. Incorporating different methodologies different multiple model switching control techniques can be developed. One of the key issue in multiple model control is the design of members of the model bank and the design of a proper switching mechanism [1, 2]. In many papers, this point was discussed and different solutions were provided for designing models of the original system [2–4]. A multiple model controller which combines adaptive neuro-fuzzy inference system with multiple models was proposed [5]. As an example of controller design in multiple model control, $H_{\infty}$ controller can be given [6, 7]. Different strategies have been followed in the design and adaptation of models of multiple model control scheme i.e. fixed and adaptive identification models was studied [8].
Modal control which is classified in active vibration control [9, 10] has been studied intensively, in recent years. Independent modal space control was proposed [11] to minimize an infinite horizon quadratic cost function representing the sum of kinetic energy in each mode shape. Efficient modal control strategy where optimal control theory was used to determine the modal control gains, was studied [12, 13]. In early studies in this fields, single mode shape of motion was controlled [14, 15]. Later, multi-modal control of structural systems was studied [16]. In multi-modal control, timing was determined by a control scheme that observes the rate of energy change in controlled modes [16]. Active modal control of significant mode shapes of a structural system with an observer was studied [17, 18]. In general, finite number of mode shapes of a structural system is split into controlled and uncontrolled mode shapes: modal control has been used to control only those mode shapes called the controlled mode shape.

In this article, a novel multiple model controller scheme which is composed of approximate dynamic sub-models of a structural system, was proposed. The modal controllers in the controller bank of the switching multiple model control were associated with each dynamic sub-models. In the proposed multiple model control scheme, the modal control in effect changes as the plant model switches among dynamic sub-models of the plant. However, those modal control schemes which were proposed in the past, are lacking of a switching mechanism that is an inherent part of the multiple model control theory. In modal control theory and its application to the control of mode shapes of structural systems, the set of mode shapes that are controlled is fixed [19]. From a model set point of comparison, it can be said that those applications of the multiple model control schemes are lacking of the use of approximate dynamic sub-models. Switching multiple model control theory [20] has been primarily used to control nonlinear systems which are approximated by rather simple models i.e approximate linear time-invariant model of the nonlinear system.

With respect to the contemporary literature on optimal control and multiple model control, this study has made the following contributions: a novel multiple model control scheme was proposed to control structural systems. This paper has shown that switching multiple model control can be used to control structural systems. This might have useful consequences when the control signal power supply has limited bandwidth and/or amplitude/rate capacity.

III. PROBLEM STATEMENT

In the switching multiple model control scheme, it is assumed that the system is described by a set of differential equations of the form [21, 22]:

\[
\dot{x}(t) = A_i x(t) + B_i z(t); \quad \forall i 
\]

(1)

Different system dynamics can be expressed by different values of \(i\). The dynamic sub-model \(M_i\) assumed to be fixed in different environments represented by different matrices \((A_i, B_i)\). The supervisor is made of a bank of dynamic sub-models \(M_1, M_2, ..., M_n\) where each one represents a possible outcome of the plant dynamics such that the controller \(C_i\) stabilizes and meets the desired performance specifications when it is applied to dynamic sub-model \(M_i\). The role of the supervisor is to select the controller that is actually connected to the plant at a given instant of time, so that the best performance is achieved.

In the switching multiple model control scheme proposed here, the bank of dynamic sub-models i.e. \(M_i\) are designed as a single-degree-of-freedom dynamic sub-model with accompanying mode shape and a modal controller \(C_i\) which regulates only a single dynamic sub-model that is attached to it. At any time instant, the switching multiple model control detects the most active dynamic sub-model i.e. \(M_i\) and switches to a particular modal controller \(C_i\) that regulates only the dynamic sub-model of concern.

IV. SWITCHING MULTIPLE MODAL CONTROL SYNTHESIS FOR STRUCTURAL SYSTEMS

A. Modal Control

Dynamics of a damped structural system having \(n\) degree-of-freedom is governed by the equation:

\[
M \ddot{q} + D \dot{q} + K q = L_f z 
\]

(2)

where the proportional damping matrix is defined as: \(D = \alpha M + \beta K\). \(\alpha\) and \(\beta\) are some scalar constants that multiply mass and stiffness matrices. \(M \in \mathbb{R}^{n \times n}\) is the mass matrix. \(D \in \mathbb{R}^{n \times n}\) is the proportional damping matrix. \(K \in \mathbb{R}^{n \times n}\) is the stiffness matrix. \(q \in \mathbb{R}^n\) is the vector of generalized coordinates. \(z \in R\) is the excitation force. \(L_f \in \mathbb{R}^n\) is the input signal influence vector. Let us consider the modal matrix \(\Phi\). When mode shapes are normalized with respect to the mass matrix, it can be shown that mode shapes have the following orthogonality properties with respect mass matrix \(M\) and stiffness matrix \(K\):

\[
\phi_i^T M \phi_j = \delta_{ij} \quad \phi_i^T K \phi_j = \omega_i^2 \delta_{ij}; \quad \forall i, j
\]

(3)

Using the following linear transformation

\[
q = \Phi \eta
\]

(4)
in Eq. (2) and premultiplying by \(\Phi^T\). Using the orthogonality property presented in Eq. (3), it can be shown this leads to
\[ \ddot{\eta}_i + 2\zeta_i \omega_i \dot{\eta}_i + \omega_i^2 \eta_i = \phi_i^T L_f z; \quad \forall i \quad (5) \]

The total solution can be written as a linear summation of mode shapes multiplied by principal coordinates which are calculated by solving separate differential equations given in Eq. (5).

\[ q = \sum_{i} \phi_i \eta_i \quad (6) \]

The uncoupled differential equations can be represented in state-space formulations as given in Eq. (1). Here the state matrix is denoted by \( A_i \in R^{2 \times 2} \); the control signal gain vector is denoted by \( B_i \in R^2 \); the state vector is denoted by \( x_i \in R^2 \); the control signal is denoted by \( z \in R \).

\[ A_i = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta_i \omega_i \end{bmatrix}; \quad B_i = \begin{bmatrix} 0 \\ \phi_i^T L_f \end{bmatrix}; \quad x_i = \begin{bmatrix} \eta_i \\ \dot{\eta}_i \end{bmatrix} \quad (7) \]

State-feedback signal is generated for the \( i \)-th dynamic sub-model as follows:

\[ z_i = -K_i x_i \quad (8) \]

Using Eq. (6) and the orthogonality of mode shapes with respect to mass matrix given in Eq. (3), \( x_i \) can be recovered from \( q \) and \( \dot{q} \) with following equation

\[ x_i = \begin{bmatrix} \phi_i^T M q \\ \phi_i^T M \dot{q} \end{bmatrix} \quad (9) \]

It can be shown that poles of the \( i \)-th dynamic sub-model can be displaced without disturbing the poles of the remaining \( j \)-th dynamic sub-models \( \forall j \neq i \). This shows that the modal control signal can change the damping ratio and natural frequency of a single dynamic sub-model i.e. its poles. Hence single dynamic sub-model can be controlled one at a time. The set of controllers i.e. \( C_i \) in Eq. (1) for multiple model control scheme can be designed by using Eq. (8) as separate state-feedback controllers i.e. modal controllers.

### B. Multiple Model Representation of a Structural System

Using the linear transformation defined in Eq. (4) it can be shown that Eq. (2) is transformed into the following state-space representation

\[ \dot{x} = Ax + Bz \quad (10) \]

\[ A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix}; \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} \quad (11) \]

\[ x = \begin{bmatrix} x_1^T \\ x_2^T \\ \cdots \\ x_n^T \end{bmatrix} \quad (12) \]

Matrices and vectors in Eq. (10-12) were defined in previous pages. The output signal is defined as: \( y = Cx \). The total
The response of the linear time-invariant structural system can be written as a linear combination of mode shapes; see Eq. (6). The set of models i.e. \( M_i \) in Eq. (1) can be defined by using Eq. (1.7,10-12). It can be shown that each \( M_i \) is represented by a dynamic sub-model as given in Eq. (1.7). The dynamic sub-model are represented as follows:

\[
M_i = \{ \frac{\ddot{y}_i}{t} = 2\zeta \omega_n \frac{\dot{y}_i}{t} + \omega_n^2 \frac{y_i}{t} = b_i z_i \} \quad (i \in N')
\]

If one selects a single dynamic sub-model which resembles the total response the most then one can claim that the system switches among members of the set of dynamic sub-models: \( \Sigma_{i \in N'} \{ M_i \} \). \( N' \) is the set of dynamic sub-models. According to the output model of the original system, for each dynamic sub-model an output model is defined as: \( y_i = C_i (E_i x_i) \). \( C_i \) is a suitable output matrix. \( E_i \) is a matrix that transforms output from principal coordinates to generalized coordinates for sub-model \( i \).

C. Switching Multiple Modal Control Synthesis

Although each dynamic sub-model participates in the total response, it is quite obvious that some dynamic sub-models would be dominant. If one selects a single dynamic sub-model that resembles the total response the most, one can claim that the system switches among members of the set of dynamic sub-models. The set of models of the switching multiple model control scheme is formed from the dynamic sub-models i.e. \( M_i \). Dynamic sub-model of the plant, \( M_i \) approximates the output of plant dynamics as follows:

\[
\{ M_i \Rightarrow q_i(t) \approx \phi_i \eta_i(t) \} 
\]

In order to initiate the multiple model control scheme, a switching mechanism is needed to identify the dynamic sub-model that participates in the system output more than other dynamics sub-models. This switching mechanism was transformed into the cost-to-go to return to the origin. It is expected that those dynamic sub-models demanding more energy would participate in the system response the most. Hence, a novel switching mechanism was developed by considering an infinite horizon quadratic cost function for each uncontrolled dynamic sub-model. Then the Lyapunov equation was solved to determine the relative cost-to-go to return to the origin and the dynamic sub-model that requires the highest cost-to-go to return to the origin was selected and the modal controller was applied to control that dynamic sub-model. Details of the novel switching mechanism are presented next: if there is no control signal applied on any sub-models then state-space representation of each sub-model becomes

\[
x_i = A_i x_i; \quad y_i = C_i (E_i x_i)
\]

Then an infinite horizon cost function was defined for the output model. Using the output model given in Eq. (15), this cost function was transformed to

\[
J_i^A = \frac{1}{2} \int_0^\infty (x_i^T Q_i^R x_i) dt
\]

The new penalty matrix is defined as: \( Q_i^L = [C_i E_i]^T Q_i^R [C_i E_i] \). The value of infinite horizon cost function is calculated as: \( J_i^L = x_i^T(0) P_i^R x_i(0) \). \( P_i^R \) can be calculated by solving the Lyapunov equation. The optimization problem for switching modal control method that defines the optimal switching rule is then expressed as:

\[
\arg\max_{(i, M_i)} J_i^A \Rightarrow z^o = z_i
\]

This optimization problem compares the value of infinite horizon cost function defined for each dynamic sub-model. For dynamic sub-model having high energy, it is expected that the cost function takes high values. On the contrary, for dynamic sub-model that dissipates energy at high rates, this cost function takes small values. Therefore, by comparing the cost function for each dynamic sub-models, it is possible to identify the most active dynamic sub-model at any time. Hence, a proper state-feedback controller can be selected to regulate the most active dynamic sub-model using the modal controller designed by linear quadratic regulator.

In order to decide on the optimal amplifier gains as well as the optimal eigenvalues of the dynamic sub-model of concern, an infinite horizon quadratic cost function was considered for the dynamic sub-model \( M_i \). Using the output model, this cost function was transformed to

\[
J_i^R = \frac{1}{2} \int_0^\infty (x_i^T Q_i^R x_i + z_i^T R z_i) dt
\]

V. SIMULATIONS

In this section the switching modal control scheme proposed in this paper was applied to the problem of a rotor-disk system shown in Fig. 1. First, a finite element model of the system was built in ANSYS Workbench 2020R2. The Rayleigh damping model was used to account for the energy dissipation elements. In order to reduce the size of finite element model from \( m \times m \) to \( 3 \times 3 \) modal reduction was used. The first four mode shape of the rotor-disk system calculated by ANSYS workbench 2020R2 are shown in Fig. (2-5). The natural frequency for the first four mode shapes are: \( \omega_n = \{ 0.0055; 36.2338; 63.5167; 188.7369 \} \) Hz. Hence, by using the first/second and third mode shape and the modal reduction technique a three degree of freedom discrete system was built. The subset of mode shapes were selected as: \( q = [\Phi_r \eta_r \Phi_r] = [\phi_1 \phi_2 \phi_3] \).

Using the modal reduction technique and the reduced-order modal matrix \( \Phi_r \), the finite element model was transformed into form represented by Eq. (10-12). This state-space representation has three independent dynamic sub-model associated with each mode of motion \( \{ \Phi_r, \omega_n \} \). Then for each dynamic
model in the form given in Eq. (1,7), a modal controller was designed using Eq. (8,18). The cost-to-go to return to the origin for each dynamic sub-model was calculated with Eq. (16). To simulate the switching optimal modal control of the rotor-disk system the optimization problem defined in Eq. (17) was solved at every time step.

In Fig 6-7, Bode diagram of the complete reduced-order torsional model is compared with the Bode diagram of dynamic sub-models. It is apparent that each dynamic sub-model is dominant in a certain frequency range centered around a natural frequency. The system was simulated on MATLAB Simulink program with parameters: numerical method: ode4 Runge-Kutta with fixed step size of $1 \times 10^{-3}$ sec; simula-
tion time 60 seconds. Control signal was applied on disk four and disturbance signal was applied on disk one and angular displacement of third disk was controlled. Simulation parameters are: $\alpha = 1 \times 10^{-2}$; $\beta = 1 \times 10^{-12}$; $Q^R_i = \text{diag} \{1 \times 10^{-1}, 1 \times 10^0\}$; $Q^L_i = \text{diag} \{1 \times 10^1, 1 \times 10^3\}$; $r^R_i = 1 \times 10^{-1}$ and the total disturbance force acting on the rim of disk-one has magnitude of $F_4 = 1000N$. In the simulation, switching modal control scheme designed by Eq. (18) i.e. Lqr method was applied on fourth disk to control the angular displacement of third disk. The resulting switching rule is shown in Fig. 8. The disturbance signal acting on dynamic sub-model 1/2/3 are shown in Fig. 9. The control signal acting on dynamic sub-model 1/2/3 are shown in Fig. 10. Fig. 11-14 show the time history of angular displacement of all disks. These results shows that switching modal controller regulated the system successfully.

VI. CONCLUSIONS

Frequency sweep analysis of a structural system has shown that when the external excitation is in the neighborhood of any natural frequency, a single-degree-of-freedom dynamic sub-model is sufficient to represent the system response with a reasonable error. This observation has been used to develop a switching multiple model approximation of the original structural system. For each uncontrolled dynamic sub-model the cost-to-go to return to the origin was calculated by solving Lyapunov equation. Next, a simple optimization problem was solved by comparing these relative cost-to-go to return to the origin and the dynamic sub-model having the highest cost was picked as the most active dynamic sub-model. This dynamic sub-model was regulated by the associated modal controller. Thus it has been shown that a multiple model control scheme can be designed for a structural system. It has been shown that

\footnote{Matlab software that simulates this numerical case study can be downloaded from MATLAB Central File Exchange: https://www.mathworks.com/matlabcentral/fileexchange/97839-switching-optimal-modal-control}
this controller targets the most active dynamic sub-model that participates in the system response more than the remaining dynamic sub-models. It is believed that this multiple model switching controller scheme can use the power source of control signal effectively when the amplitude/rate of control signal and the bandwidth of the power source is limited.

REFERENCES


